A Proof of the Hodge Conjecture

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1 Introduction

In this paper we show that the Hodge conjecture and a part of the Tate conjecture hold. Since it is difficult to find algebraic cycles in general, the strategy is to proceed by induction argument to vanish a certain subspace of a cohomology of an open affine subvariety of an affine variety which is obtained excluding a general hyperplane from a given variety.

2 In Case of Non Singular Varieties

Let \( k \) be a field with an algebraic closure \( \bar{k} \) and \( X \) a smooth geometrically irreducible variety over \( k \). There exists the canonical cycle map for \( \ell \neq \text{char} \ k \)

\[
cl_\ell^r : CH^r(X) \rightarrow H^{2r}_{\text{et}}(X_k, \mathbb{Q}_\ell(r))
\]

This image is included in the fixed part

\[
H^{2r}_{\text{et}}(X_k, \mathbb{Q}_\ell(r))^{G_k}
\]

where \( G_k = \text{Gal}(\bar{k}/k) \). Tate's conjecture says that if \( k \) is finitely generated as a field, the image of \( cl_\ell^r \) generates \( H^{2r}_{\text{et}}(X_k, \mathbb{Q}_\ell(r))^{G_k} \). Fix an isomorphism \( \iota : \mathbb{Q}_\ell \rightarrow \mathbb{C} \).

Let \( k = \mathbb{C} \). One has the canonical cycle map

\[
cl^r : CH^r(X) \rightarrow H^{2r}(X(\mathbb{C}), \mathbb{Q}(2\pi i)^r)
\]

This image is included into

\[
H^{2r}(X(\mathbb{C}), \mathbb{Q}(2\pi i)^r) \cap H^r(\mathbb{C})
\]

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Hodge conjecture says that the image of $\text{cl}^r$ generates $H^{2r}(X(C), Q(2\pi i)^r) \cap H^{r\cdot r}(X(C))$. Let $U$ be a smooth quasiprojective variety over $k$. The images of the canonical cycle maps are

$$\begin{cases} H^{2r}_d(U_k, Q(r))^{G_k} & \text{for finitely generated } k \\ F^r H^{2r}(U, C) \cap W_{2r} H^{2r}(U, Q(r)) & \text{for } k = C \end{cases}$$

Let $U$ be a smooth quasi-projective variety over $k$ and $X$ a smooth projective compactification of $U$. One denotes by $\text{cl}^*$ the following cycle maps $\text{cl}_{DR}$, $\text{cl}_\ell$, $\text{cl}_H$;

(a) $\Gamma_{DR}(H^{2r}_{DR}(U)(r)) = W_0(H^{2r}_{DR}(U)(r)) \cap F^0(H^{2r}_{DR}(U)(r))$

(b) $\Gamma_{\ell}(H^{2r}_{\ell}(U)(r)) = H^{2r}_{\ell}(U)(r)^{G_k} \cap W_0(H^{2r}_{\ell}(U)(r))$

(c) $\Gamma_{H}(H^{2r}_H(U)(r)) = W_0(H^{2r}_H(U)(r)) \cap F^0(H^{2r}_H(U)(r)) \otimes C$

Lemma 1 It suffices to prove it for a smooth affine variety over $k$.

Proof. It is well known that it is enough to treat it in the case of $\dim X = 2d$. Choose a smooth irreducible hyperplane $Y$ on $X$. Thus $X - Y$ is a smooth affine variety. One obtains the following commutative diagram:

$\begin{array}{ccc}
\text{CH}^{d-1}(Y) & \longrightarrow & \Gamma_* H^{2d}_{s,Y}(X)(d) \\
\downarrow & & \downarrow \\
\text{CH}^{d}(X) & \longrightarrow & \Gamma_* H^{2d}(X)(d) \\
\downarrow & & \downarrow \\
\text{CH}^{d}(X - Y) & \longrightarrow & \Gamma_* H^{2d}(X - Y)(d) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$

Here the vertical sequences are exact. By duality, one has $H^{2d}_{s,Y}(X)(d) \cong H^{2d-2}(Y)(d - 1)$. Assume conjectures hold for $X - Y$. Induction hypothesis for $\text{CH}^d(Y) \longrightarrow \Gamma_* H^{2d-2}(Y)(d - 1)$ implies conjectures.

Lemma 2 The action of $G$ on $M$ keeps $B$ to be invariant.
Proof. Since $A$ is regular; hence normal, $A$ coincides with the integral closure of $k[x_1, \cdots, x_{2d}]$ in $L$. Let $a \in A$ with the minimal polynomial hand let $\sigma \in G$. Then $h^\sigma = h$. Take any $b \in B$, which satisfies the minimal polynomial $g(b) = 0$ with coefficients in $A$. Thus $g^\sigma$ has its coefficients in $A$. Hence $g^\sigma(b^\sigma) = 0$, which implies $b$ is an integral element of $M$. Therefore $b^\sigma \in B$. ■

Let $K'$ be the field of the invariants of $M$ by $G$ and $C$ the integral closure of $k[x_1, \cdots, x_{2d}]$. Note that $K'$ is a radical extension of $K$ of finite degree.

Lemma 3 (i) $B^G = C$

(ii) $\text{Spec } C \to \text{Spec } k[x_1, \cdots, x_{2d}]$ is a finite, surjective and radical morphism, which is a universal homeomorphism.

Proof. Since $M \supset B \supset A \supset C \supset k[x_1, \cdots, x_{2d}]$, one has $K' = M^G \supset B^G \supset A^G \supset C^G = C$. Since $B$ is a finite $k[x_1, \cdots, x_{2d}]$ module and $k[x_1, \cdots, x_{2d}]$ is a Noetherian ring, every submodule of $B$ is a finite $k[x_1, \cdots, x_{2d}]$ module. Every element of $B^G$ is integral over $C$. Hence $B^G = C$ since $C$ is integrally closed. $\phi : \text{Spec } C \to \text{Spec } k[x_1, \cdots, x_{2d}]$ For every point $x$ of $\text{Spec } C$ one has $\kappa(x)$ is a radical extension of $\kappa(x)$; thus $\phi : \text{Spec } C \to \text{Spec } k[x_1, \cdots, x_{2d}]$ is a radical morphism. It is clear that the morphism is finite and surjective; hence a universal homeomorphism. ■

Let $\mathcal{F}$ be a suitable smooth sheaf on $X$. One has a trace map: $\text{tr}_{L/K'} : L \to K'$, which naturally extends to a map $\text{tr}_{A/C} : A \to C$. If $f \in C$, one further has a map $\text{tr}_{A[1/\mathcal{F}]/C[1/\mathcal{F}]} : A[1/\mathcal{F}] \to C[1/\mathcal{F}]$ and a cohomological map

$$\text{tr}_{A[1/\mathcal{F}]/C[1/\mathcal{F}]} : H^i \left( \text{Spec } A[1/\mathcal{F}], \mathcal{F} \right) \to H^i \left( \text{Spec } C[1/\mathcal{F}], \mathcal{F} \right).$$

If $f \in k[x_1, \cdots, x_{2d}]$, $\phi : \text{Spec } C[1/\mathcal{F}] \to \text{Spec } k[x_1, \cdots, x_{2d}, 1/\mathcal{F}]$ is a finite, surjective and radical morphism; hence a universal homeomorphism. Thus one has an isomorphism

$$H^i \left( \text{Spec } C[1/\mathcal{F}], \mathcal{F} \right) \to H^i \left( \text{Spec } k[x_1, \cdots, x_{2d}, 1/\mathcal{F}], \mathcal{F} \right).$$

Let $H = \text{Gal}(M/L)$.

Lemma 4 Assume that $\dim \text{Spec } A = n = 2d \geq 4$. Let $f_1, \cdots, f_n$ be $k$-linear combinations of $x_1, \cdots, x_n$ in $k[x_1, \cdots, x_n]$. Assume that the intersection locus $V(f_1, \cdots, f_n)$ is void in $\text{Spec } k[x_1, \cdots, x_n]$, i.e., the hyperplanes intersect one point in infinity. One obtains $\Gamma_x, H^n \left( \text{Spec } A[1/\mathcal{F}], \mathcal{F} \right) = 0$.

Proof. If $n > 2$, by the affine vanishing theorem, one has $H^{2n-1} \left( \text{Spec } A, \mathcal{F} \right) = 0, H^{2n-2} \left( \text{Spec } A, \mathcal{F} \right) = 0.$
One has a Čech Spectral sequence:

\[ E_1^{p,q} = \oplus_{|I|=p+1} H^q \left( \cap_{i \in I} \text{Spec } A[\frac{1}{f_i}], \mathcal{F} \right) \to H^{p+q} \left( \text{Spec } A - V(f_1, \ldots, f_n), \mathcal{F} \right). \]

Note that \( E_1^{pq} = 0 \) for \( q > n \) by affine vanishing theorem and that \( p > n - 1 \) by Čech cohomology theory. One obtains

\[ E_{\infty}^{n-1,n} = H^{2n-1} \left( \text{Spec } A - V(f_1, \ldots, f_n), \mathcal{F} \right) = 0, \]

\[ E_2^{n-2,n} = E_{\infty}^{n-2,n} = Gr^{n-2}H^{2n-2}((\text{Spec } A, \mathcal{F}) = 0, \]

since \( E_2^{n-4,n+1} = E_2^{n-1,n-1} = 0 \). Note that there exists the following exact sequence:

\[ \Gamma_* H_2^{d-2}(V(f_i), \mathcal{F})(d-1) \rightarrow \Gamma_* H_2^{2d}(\text{Spec } A[\frac{1}{f_1 \cdots f_k}], \mathcal{F})(d) \rightarrow \Gamma_* H_2^{2d}(\text{Spec } A[\frac{1}{f_1 \cdots f_k}], \mathcal{F})(d) \rightarrow 0. \]

Thus,

\[ \oplus_{|I| = n-2} \Gamma_* H^n \left( \text{Spec } A[\frac{1}{f_1 \cdots f_k}], \mathcal{F} \right) \to \oplus_{|I| = n-1} \Gamma_* H^n \left( \text{Spec } A[\frac{1}{f_1 \cdots f_k}], \mathcal{F} \right) \]

and

\[ \oplus_{|I| = n-1} \Gamma_* H^n \left( \text{Spec } A[\frac{1}{f_1 \cdots f_k}], \mathcal{F} \right) \to \oplus_{|I| = n} \Gamma_* H^n \left( \text{Spec } A[\frac{1}{f_1 \cdots f_k}], \mathcal{F} \right) \]

are surjections, respectively. Using \( E_2^{n-3,n+1} = E_2^{n+1,n-1} = E_1^{n,n} = 0 \), one has

\[ E_3^{n-1,n} = H \left( E_2^{n-3,n+1} \to E_2^{n-1,n} \to E_2^{n+1,n-1} \right) = \]

\[ E_2^{n-1,n} = H \left( E_1^{n-2,n} \to E_1^{n-1,n} \to E_1^{n,n} \right) = E_{\infty}^{n-1,n} = 0. \]

Hence, the homomorphism

\[ E_1^{n-2,n} \to E_1^{n-1,n} \]

is a surjection.

Applying \( E_2^{n-4,n+1} = E_2^{n-1,n} = 0 \), one has

\[ E_3^{n-2,n} = H \left( E_2^{n-4,n+1} \to E_2^{n-2,n} \to E_2^{n,n-1} \right) = \]
\[ E_2^{n-2,n} = H\left( E_1^{n-3,n} \to E_1^{n-2,n} \to E_1^{n-1,n} \right) = E_\infty^{n-2,n} = 0. \]

Moreover, since
\[ E_2^{n-2,n} = H\left( \bigoplus_{|I|=n-2} H^n(\text{Spec } A[\prod_{i \in I} f_i]), \mathcal{F}) \to \bigoplus_{|I|=n} H^n(\text{Spec } A[\prod_{i \in I} f_i]), \mathcal{F}) \right) = 0 \]
and the homomorphism
\[ \bigoplus_{|I|=n-1} \Gamma_* H^n(\text{Spec } A[\prod_{i \in I} f_i]), \mathcal{F}) \to \bigoplus_{|I|=n-1} \Gamma_* H^n(\text{Spec } A[\prod_{i \in I} f_i]), \mathcal{F}) \]
is surjective and a functor \( \Gamma_* \) is exact, it implies that
\[ \bigoplus_{|I|=n-1} \Gamma_* H^n(\text{Spec } A[\prod_{i \in I} f_i]), \mathcal{F}) \to \bigoplus_{|I|=n} \Gamma_* H^n(\text{Spec } A[\prod_{i \in I} f_i]), \mathcal{F}) \]
is a zero map.

On the other hand,
\[ \bigoplus_{|I|=n-1} \Gamma_* H^n(\text{Spec } A[\prod_{i \in I} f_i]), \mathcal{F}) \to \bigoplus_{|I|=n} \Gamma_* H^n(\text{Spec } A[\prod_{i \in I} f_i]), \mathcal{F}) \]
is a surjection. Thus one concludes that
\[ \Gamma_* H^n(\text{Spec } A[\prod_{i=1}^n f_i], \mathcal{F}) = \bigoplus_{|I|=n} \Gamma_* H^n(\text{Spec } A[\prod_{i \in I} f_i]), \mathcal{F}) = 0. \]

**Theorem 5** For \( 0 \leq k \leq n \) the images of
\[ \text{CH}^d(\text{Spec } A)[\prod_{i=1}^k f_i]) \to \Gamma_* H^{2d}(\text{Spec } A[\prod_{i=1}^k f_i]) \]
generate the targets. In particular, the image of
\[ \text{CH}^d(\text{Spec } A) \to \Gamma_* H^{2d}(\text{Spec } A) \]
generates the target.

**Proof.** One continues to proceed by induction argument: \( H^n_{V(f_i)} (\text{Spec } A, \mathcal{F}) \to H^n (\text{Spec } A, \mathcal{F}) \to H^n (\text{Spec } A[\prod_{i=1}^k f_i], \mathcal{F}) \)
\[ \ldots \]
H^n_{V(f_n)} (\text{Spec} \ A[\frac{1}{t_1}, \ldots, \frac{1}{t_{n-1}}], \mathcal{F}) \rightarrow H^n (\text{Spec} \ A[\frac{1}{t_1}, \ldots, \frac{1}{t_{n-1}}], \mathcal{F}) \rightarrow H^n (\text{Spec} \ A[\frac{1}{t_1}, \ldots, \frac{1}{t_n}], \mathcal{F})

One has the following commutative diagram, whose vertical sequences are exact:

\[
\begin{array}{ccc}
\text{CH}^{d-1}(V(f_k)) & \rightarrow & \Gamma_s H^{2d-2}(V(f_k))(d-1) \\
\downarrow & & \downarrow \\
\text{CH}^d(\text{Spec} \ A[\frac{1}{t_1}, \ldots, \frac{1}{t_{k-1}}]) & \rightarrow & \Gamma_s H^{2d}(\text{Spec} \ A[\frac{1}{t_1}, \ldots, \frac{1}{t_{k-1}}])(d) \\
\downarrow & & \downarrow \\
\text{CH}^d(\text{Spec} \ A[\frac{1}{t_1}, \ldots, \frac{1}{t_k}]) & \rightarrow & \Gamma_s H^{2d}(\text{Spec} \ A[\frac{1}{t_1}, \ldots, \frac{1}{t_k}])(d) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

\[\blacksquare\]

**In Case of Singular Varieties**

Uwe Jannsen proved that the Hodge conjecture and the Tate conjecture for singular varieties are deduced by the original conjectures. For the readers convenience we explain it. For a smooth variety $X$ of dimension $d$ one has the Poincaré duality $H_{2d}(X, i) \cong H^{2d-2i}(X, d-i)$. There is no such duality in general for non smooth varieties. Fundamental classes induce a cycle map: $\text{cl}_i : Z_i(X) \rightarrow H_{2i}(X, i)$, which factors through the canonical cycle map that Fulton defines $\text{CH}_i(X) \rightarrow H_{2i}(X, i)$. The Hodge conjecture for singular varieties says that for all $i \geq 0$ the map

\[\text{cl}_i \otimes \mathbb{Q} : Z_i(X) \otimes \mathbb{Q} \rightarrow \Gamma_s H_{2i}(X, \mathbb{Q})(i) = (2\pi i)^{-i} W_{-2i} H_{2i}(X, \mathbb{Q}) \cap F^{-i} H_{2i}(X, \mathbb{C})\]

is surjective.

**Theorem 6** The Hodge conjecture is true for singular varieties.

**Proof.** By Chow’s lemma and Hironaka’s resolution of singularities for a singular non complete variety $X$ there exist $\pi : X' \rightarrow X$ a projective and surjective morphism with $X'$ quasi-projective and smooth and $\alpha : X' \rightarrow X''$ an open immersion with $X''$ projective and smooth forming a diagram of varieties

\[
\begin{array}{ccc}
X' & \xrightarrow{\alpha} & X'' \\
\downarrow & & \downarrow \\
X & & \\
\end{array}
\]

\[
\begin{array}{ccc}
Z_i(X'') \otimes F & \xrightarrow{\text{cl}_i} & Z_i(X') \otimes F \\
\downarrow & & \downarrow \\
\Gamma H_{2i}(X'', i) & \xrightarrow{\Gamma \alpha_*} & \Gamma W_0 H_{2i}(X', i) \\
\end{array}
\]

\[\blacksquare\]
Since $H_2(X', i)$ is a semi-simple object, $W_0H_2(X, i)$ is a direct factor of $H_2(X', i)$ via $\pi_* \circ \alpha^*$ and so $\Gamma\pi_* \circ \Gamma\alpha^*$ is surjective. Thus $Z_i(X) \otimes F \to \Gamma W_0H_2(X, i)$ is surjective.

Appendix

Theorem 7 Let $f_0 : X_0 \to Y_0$ a projective morphism, $\ell \in H^2(X_0, Q\ell(1))$ the first Chern class of an $f_0$-ample invertible sheaf and $F_0$ a perverse sheaf over $X_0$ for $i \geq 0$. The following map is an isomorphism

$$\ell^i : \mathcal{H}_{c-2} f_* F_0 \cong \mathcal{H}_c f_* F_0(i)$$

Lemma 8 It suffices to prove the Hodge conjecture in case of $i = 2d = \dim X$.

Proof. By the strong Lefschetz theorem it reduces to the case $i = 2p > 2d$. Let $Y$ be a general hyperplane section of $X$. By the weak Lefschetz one has an exact sequence $H^{i-2}(Y, \mathcal{F}) \to H^i(X, \mathcal{F}) \to H^i(X - Y, \mathcal{F}) = 0$. The following commutative diagram completes the proof:

\[
\begin{array}{ccc}
\text{CH}^{p-1}(Y) & \longrightarrow & \Gamma_* H^{2p-2}(Y)(p-1) \\
\downarrow & & \downarrow \\
\text{CH}^p(X) & \longrightarrow & \Gamma_* H^{2p}(X)(p) \\
\downarrow & & \downarrow \\
\text{CH}^p(X - Y) & \longrightarrow & \Gamma_* H^{2p}(X - Y)(p) \\
\downarrow & & \downarrow \\
0 & & 0 \\
\end{array}
\]

Theorem 9 Let $f : X \to Y$ be an affine morphism. The functor

$$Rf_* : D^b_c(X, \overline{\mathbb{Q}}_\ell) \to D^b_c(Y, \overline{\mathbb{Q}}_\ell)$$

is right $t$-exact. In particular, Let $k$ be an algebraically closed field and $\mathcal{F}$ an etale sheaf on $X$. $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$.

References


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